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Stagnation Point Boundary Layer on a Subliming Surface with Numerous Small Roughness Elements

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Introduction

LIBBY[1] presents an analysis of the laminar boundary layer at a two-dimensional stagnation point on a subliming surface which has a structure such that roughness elements result from the recession. The basic properties of the boundary layer depend in a fundamental way on a nondimensional parameter Γ which is proportional to n, the number of roughness elements per unit area. The problem is formulated so that as n and thus Γ increase, the extent of the roughness elements relative to the boundary-layer thickness decreases. When $\Gamma \sim \infty$ the boundary layer behaves as though slip occurs at the surface. Libby[1] demonstrates this result by means of an asymptotic analysis of a model problem obtained by neglecting the drag of the roughness elements; the outer solution is given by the usual stagnation point equations with modified boundary conditions at the wall and is shown to compare well with the complete solution of the equations involving roughness for $\Gamma = 10$. The purpose of this Note is to carry out the analysis of the complete asymptotic problem for constant density flow including the drag of the roughness elements.

Solution Methods

We begin by considering the streamwise momentum equation of Ref. 1 for the case of incompressible flow, namely

$$[(1-\alpha)F']' + (f-\alpha f_w)F' + (1-\alpha)(1-F^2)$$
$$+\Gamma(2\alpha^{1/2}f_w - \alpha k)F = 0 \qquad (1)$$

where

$$F = f'(\eta)/(1-\alpha) \qquad (2)$$

$$\alpha = 1 \qquad \eta \le 0 \qquad (3a)$$

$$\alpha = (1-\Gamma\eta)^2 \qquad 0 \le \eta \le \Gamma^{-1} \qquad (3b)$$

$$\alpha = 0 \qquad \eta \ge \Gamma^{-1} \qquad (3c)$$

with the associated boundary conditions

$$\eta \to \infty, \quad f' = 1; \quad \eta = 0, \quad f - f_w = f' = 0 \qquad (4)$$

Primes denote differentiation with respect to the similarity variable η defined, along with the stream-function f, in standard form in Ref. 1. The specified parameter f_w represents the injection rate associated with sublimation, and k (in Libby's notation $= m_D \rho_S v_S / 2r$) is a measure of the drag on the fluid passing around the roughness elements. The quantity $\alpha(\eta)$ is the fraction of area covered by solid.

Now we define $\epsilon = \Gamma^{-1}$ and develop solutions in the limit $\epsilon \to 0$ for $f_w = 0(1)$ and $k = 0(1)$. In the outer region, defined by Eq. (3c), Eq. (1) and the first part of Eq. (4) become

$$f''' + ff'' + (1 - f'^2) = 0; \quad \eta \to \infty, \quad f' = 1 \qquad (5)$$

If we assume that $f \sim f_0 + \epsilon f_1 + \ldots$, then

$$f''' + f_0 f_0'' + (1 - f_0'^2) = 0; \quad \eta \to \infty, \quad f_0' = 1 \qquad (6)$$

$$f_1''' + f_0 f_1'' - \alpha f_0' f_1' + f_1 = 0; \quad \eta \to \infty, \quad f_1' = 0 \qquad (7)$$

Note that Eq. (6) is the usual equation for a two-dimensional stagnation point but that the solutions of interest here will involve boundary conditions at $\eta = 0$ given by matching requirements.

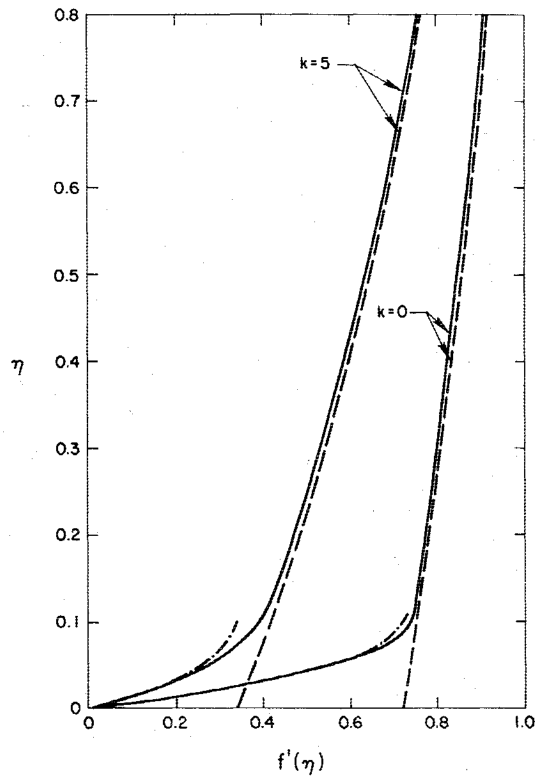

Fig. 1 A comparison of exact numerical and approximate velocity profiles for $f_w = -0.5$ and $\epsilon = 0.1$; —— $f'(\eta)$, – – – $f_0'(\eta)$, – · – · – · – $\hat{f}_i'(\hat{\eta} = \eta/\epsilon)$.

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We proceed by developing solutions in the inner region defined by Eqs. (3). If the inner variables defined by

$$\eta=\epsilon\hat{\eta},\quad f=f_w+\epsilon\hat{f}(\hat{\eta};\epsilon),\quad \hat{F}=\hat{f}'(\hat{\eta})/(1-\alpha) \tag{8}$$

are substituted into Eqs. (1-4), then we find

$$[(1-\alpha)\hat{F}']'+\epsilon\{[(1-\alpha)f_w+\epsilon\hat{f}]\hat{F}'+[2\alpha^{1/2}f_w-\alpha k]\hat{F}\}$$
$$+\epsilon^2(1-\alpha)(1-\hat{F}^2)=0 \tag{9}$$

$$\hat{f}(0)=\hat{f}'(0)=0 \tag{10}$$

where primes denote differentiation with respect to $\hat{\eta}$, $\alpha=(1-\hat{\eta})^2$, and $0\leq\hat{\eta}<1$. The integral orders of ϵ in Eq. (9) imply that the solution should be represented as $\hat{f}\sim\hat{f}_1+\epsilon\hat{f}_2+\ldots$. It follows that the describing systems are

$$\left[(1-\alpha)\left(\frac{\hat{f}_1'}{1-\alpha}\right)'\right]'=0\quad \hat{f}_1(0)=\hat{f}_1'(0)=0 \tag{11}$$

$$\left[(1-\alpha)\left(\frac{\hat{f}_2'}{1-\alpha}\right)'\right]'=-(1-\alpha)f_w\left(\frac{\hat{f}_1'}{1-\alpha}\right)'$$
$$-(2\alpha^{1/2}f_w-\alpha k)\frac{\hat{f}_1'}{(1-\alpha)}$$
$$\hat{f}_2(0)=\hat{f}_2'(0)=0 \tag{12}$$

The solution satisfying Eq. (11) is

$$\hat{f}_1=\frac{K_1}{2}\left\{(2-\hat{\eta})^2\left[\frac{(1+\hat{\eta})}{3}\ln(2-\hat{\eta})-\frac{(5+2\hat{\eta})}{18}\right]\right.$$
$$\left.+\hat{\eta}^2\left[\left(1-\frac{\hat{\eta}}{3}\right)\ln\hat{\eta}+\left(\frac{-9+2\hat{\eta}}{18}\right)\right]-\frac{4}{3}\left(\ln 2-\frac{5}{6}\right)\right\}$$
$$+L_1\hat{\eta}^2\left(1-\frac{\hat{\eta}}{3}\right) \tag{13}$$

It follows from Eq. (8) that at $\hat{\eta}=1$ or $\eta=\epsilon$, $f(\epsilon)=f_0(\epsilon)+\epsilon f_1(\epsilon)+\ldots=f_w+\epsilon\hat{f}(1;\epsilon)$. If $f_n(\eta)$, $n=0,1\ldots$ are regular functions, then $f_n(\epsilon)\sim f_n(0)+f_n'(0)\epsilon+\ldots$. To lowest order the streamfunction f is continuous at $\hat{\eta}=1$ if $f_0(0)=f_w$. A similar argument can be used to show that continuity of the velocity f' results if

$$f_0'(0)=L_1 \tag{14}$$

Finally we observe that

$$f''(\eta=\epsilon)=f_0''(0)+0(\epsilon)=\frac{1}{\epsilon}\left[\hat{f}_1''(\hat{\eta}=1)+0(\epsilon)\right] \tag{15}$$

Since we expect from Eq. (6) that $f_0''(0)=0(1)$ with respect to the limit $\epsilon\to 0$, then $\hat{f}_1''(\hat{\eta}=1)=0$ and $K_1=0$. The value of L_1 remains undefined at this point, but is given by the next higher order in the inner solution.

The solution for Eq. (12) can be written as

$$\hat{f}_2=L_1\left\{-f_w\hat{\eta}^3\left(\frac{2}{3}-\frac{\hat{\eta}}{4}\right)+\frac{k}{3}\left[\frac{\hat{\eta}^5}{10}-\frac{\hat{\eta}^4}{2}+\frac{2\hat{\eta}^3}{3}\right.\right.$$
$$\left.\left.+\frac{(2-\hat{\eta})^3}{3}\left(\ln(2-\hat{\eta})-\frac{1}{3}\right)-(2-\hat{\eta})^2\left(\ln(2-\hat{\eta})-\frac{1}{2}\right)\right]\right\}$$
$$+\frac{K_2}{2}h(\hat{\eta})+L_2\hat{\eta}^2\left(1-\frac{\hat{\eta}}{3}\right)+M_2 \tag{16}$$

where $h(\hat{\eta})$ is the functional form multiplying $(K_1/2)$ in Eq. (13). The presence of a term $0(\hat{\eta}^2\ln\hat{\eta})$ in $h(\hat{\eta})$ implies that there will be a spatial nonuniformity in the inner expansion for f constructed from Eqs. (8, 13, and 16), when $\epsilon\ln\hat{\eta}=0(1)$. This undesirable behavior will be suppressed if $K_2=0$. The boundary conditions on Eq. (12) are satisfied identically if

$$M_2=-\frac{2L_1k}{9}\left(\frac{5}{3}-2\ln 2\right)$$

Now we invoke higher order continuity of f and f' at $\eta=\epsilon$. Using the same type of argument as that given previously, we find

$$f_1(0)=-\frac{L_1}{3}\quad f_1'(0)=L_1\left(\frac{k}{6}-f_w\right)+L_2-f_0''(0) \tag{17}$$

while continuity of f'' at $\eta=\epsilon$ requires that

$$f_0''(0)=L_1\left(\frac{f}{3}-f_w\right) \tag{18}$$

Then Eqs. (14) and (18) yield

$$f_0''(0)+[f_w-(k/3)]f_0'(0)=0 \tag{19}$$

Equation (19) provides the missing boundary conditions at $\eta=0$ for Eq. (6), and corresponds to a slip condition because the shear and velocity are related. It should be noted that $f'''(\eta=\epsilon)$ is not continuous. Finally, Eq. (35) of Ref. 1 is obtained if $k=0$.

When $f_w\to 0$, the factor $(\rho_s v_s/2r)$ in the definition k is not well defined, because both the mass flux $\rho_s v_s$ and the local regression rate r vanish. The indeterminacy can be removed by defining $\eta_M\equiv\Gamma^{-1}=(\rho_s v_s/2r)(\rho_e a/\mu_e n)^{1/2}$, following Ref. 1, to find the alternative form $k=m_D\eta_M(\mu_e n/\rho_e a)^{1/2}$. The top of the roughness element corresponds to $\eta=\eta_M$. Thus in the nonsubliming case an explicit value of k can be obtained if the roughness element height η_M and the number density n are specified. Note that if k is large, then Eq. (19) implies that $f_0'(0)\ll 1$, and thus the smooth wall boundary layer is approached. However one must recognize that when $k=0(\epsilon^{-1})$ the expansion procedure described by Eqs. (9-12) fails because the k-dependent term in Eq. (9) is than 0(1).‡

Results

The $f_0(\eta)$ solution can be obtained numerically from Eqs. (6), (19), and $f_0(0)=f_w$. With $f_0'(0)$ known, L_1 is found from Eq. (14), and the lowest order inner solution from Eq. (13).

Figure 1 provides a comparison of the exact and asymptotic solutions for $f_w=-0.5$, $\Gamma=10$, and for two values of k, namely 0, 5. Shown are the exact solutions to Eq. (1), the outer solutions corresponding to $f_0'(\eta)$ satisfying the boundary condition given by Eq. (19), and the resulting inner solutions $\hat{f}_1'(\hat{\eta}=\eta/\epsilon)$ for $\epsilon=0.1$. Clearly a composite solution given by $f_0'(\eta)+\hat{f}_1'(\hat{\eta})-f'(0)$ is an good agreement with the exact solution.

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‡The authors wish to acknowledge the comment of an anonymous referee calling their attention to this point.